Unit-5

* **sclar function** : let $f(x,y,z)$ be function such that it gives us a real number (scalar) for each $(x,y,z) \in D_f$

eg : $f(x) = x$
$f(1) = 1$ (Domain of fuction (D_f)) $= R$

$f(x,y) = x^2+y^2$, $f(1,2) = 1^2+2^2 = 5$
$D_f = R^2$

* **vector valued function** : A fuction $v(x,y,z) = v_1(x,y,z)\,i + v_2(x,y,z)\,j + v_3(x,y,z)\,\hat{k}$

{where $v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)$ are sclar valued fuction of (x,y,z) is called as vector valued function.

eg. $v(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$v(1,1,1) = \hat{i} + \hat{j} + \hat{k}$

eg. $v(x,y) = xy\hat{i} + e^{x^2}y\hat{j}$

$v(1,1) = \hat{i} + e\hat{j}$

* **Derivative of vector function:**

If $\vec{v} = \vec{v}(t)$ is vector function of single variable t, the $\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} v(t+\Delta t) v$

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

⇒ **Level Surface:** Let $f(x,y,z)$ be a scalar function. if we take $f(x,y,z) = c$ [c is arbitary constant], then it is called equation of level surfaces generated by sclar function $f(x,y,z)$.

eg. $f(x,y,z) = x^2 + y^2 + z^2$

Level surfaces generated by $f(x,y,z)$ are $f(x,y,z) = c$ where c is arbitary constant.

Level Curve: If $f(x,y)$ be a scalar function of two variable x,y then, level curves generated by $f(x,y) = c$ where, c is arbitary constant.

eg. $f(x,y) = x^2 + y^2 \quad f(x,y) = c$

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Ques: find the level surfaces of the scalar field defined as

(i) $f = x + y + z$ (ii) $f = x^2 + y^2 + z^2$

(iii) $f = x^2 + y^2 - z$

Sol. (i) $f(x, y, z) = x + y + z$

So level surfaces generated by scalar function $f(x, y, z)$ are $f(x, y, z) = c$ where c is any constant.

$$\Rightarrow x + y + z = c$$

position vector

If (x, y, z) is a point the P.V of this point is written as $(x\hat{i} + y\hat{j} + z\hat{k})$. In general we denote it as $\vec{r}$ is

$$\vec{r} = \text{P.V of } (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}.$$

If $\vec{r} = \vec{r}(t)$ where t is time, then $\frac{d\vec{r}}{dt}$ = velocity vector

$$\frac{d^2 r}{dt^2} = \text{Acceleration vector.}$$

Ques: If V.P of a moving point $\vec{r}(t) = (\cos t + \sin t)\hat{i} + (\sin t - \cos t)\hat{j} + t\hat{k}$

find velocity, speed, acc^n of the particle.

Sol^n: $\vec{r}(t) = (\cos t + \sin t)\hat{i} + (\sin t - \cos t)\hat{j} + t\hat{k}$

velocity $\Rightarrow \dfrac{dr}{dt} = (-\sin t + \cos t)\hat{i} + (-\sin t + \cos t)\hat{j}$.

Acc^n: $\Rightarrow \dfrac{d^2\vec{r}}{dt^2} = (-\cos t - \sin t)\hat{i} + (-\sin t + \cos t)\hat{j}$

speed $\Rightarrow |\vec{v}| = \sqrt{(-\sin t + \cos t)^2 + (\cos t + \sin t)^2}$

$= \sqrt{1+1+1} = \sqrt{3}$.

formula's:

(i) $\dfrac{d}{dt}[\vec{c}] = 0$, where $\vec{c}$ is constant vector.

(ii) $\dfrac{d}{dt}[k\vec{f}(t)] = k\dfrac{df(t)}{dt}$, where k is scalar [constant]
$\vec{f}(t)$ is vector function.

(iii) $\dfrac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \dfrac{d\vec{u}}{dt} \pm \dfrac{d\vec{v}}{dt}$, where $\vec{v}$, v are vector function.

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(iv) $\frac{d}{dt}[\vec{u}\cdot\vec{v}] = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$

(v) $\frac{d}{dt}[\vec{u}\times\vec{v}] = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$

(vi) $\frac{d}{dt}\cancel{[\vec{v}\cdot\vec{u}\cdot\vec{w}]} =$

$\frac{d}{dt}[\vec{u}\cdot\vec{v}\cdot\vec{w}] = \left[\frac{d\vec{u}}{dt},\vec{v},\vec{w}\right] + \left[u,\frac{dv}{dt},\vec{w}\right]$

$+ \left[\vec{u}\cdot\vec{v}\cdot\frac{d\vec{w}}{dt}\right]$

(vii) $\frac{d}{dt}[\phi(t)\,\vec{u}(t)] = \phi(t)\frac{du}{dt} + \vec{u}(t)\frac{d\phi}{dt}$

(viii) $\frac{d}{dt}[\vec{u}\times(\vec{v}\times\vec{w})]$

$= \frac{d}{dt}\cancel{[\vec{u}\times(\vec{v}\times\vec{w})]}\;\frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) +$

$\vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\left(\vec{v}\times\frac{dw}{dt}\right)$

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Ques: If $\vec{u}(t) = 5t^2\hat{i} + t\hat{j} + t^3\hat{k}$

$F(t) = \sin t$, find $[f(t)\vec{u}(t)]$

Sol^n:

$$[f(t)\vec{u}(t)] = \frac{d}{dt}\left[f(t)\,\vec{u}(t)\right]$$

$$\boxed{\text{viii} : \frac{d}{dt}\left[\phi(t)\cdot\vec{u}(t)\right] = \phi(t)\frac{d\vec{u}}{dt} + \vec{u}(t)\frac{d\phi}{dt}}$$

$$= f(t)\frac{d\vec{u}}{dt} + \vec{u}(t)\frac{df}{dt}$$

$$= \sin t\left[10t\hat{i} + \hat{j} + 3t^2\hat{k}\right] + \left[5t^2\hat{i} + t\hat{j} + t^3\hat{k}\right]\cos t$$

$$= \left[10t\sin t + 5t^2\cos t\right]\hat{i} + \left[\sin t + t\cos t\right]\hat{j} + \left[3t^2\sin t + t^3\cos t\right]\hat{k}$$

or

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$$f(t)\vec{u}(t) = \sin t\left[5t^2\hat{i} + t\hat{j} + t^3\hat{k}\right]$$

$$= 5t^2\sin t\,\hat{i} + \sin t\,\hat{j} + t^3\sin t\,\hat{k}$$

$$\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = 5\left[t^2\cos t + 2t\sin t\right]\hat{i} + \left[t\cos t + \sin t\right]\hat{j} + \left[t^3\cos t + 3t^2\sin t\right]\hat{k}$$

Ques: $\vec{u}(t) = \sin(2t)\hat{i} - \cos(2t)\hat{j} + t\hat{k}$

$\vec{v}(t) = \cos 2t\,\hat{i} - \sin 2t\,\hat{j} + t^2\hat{k}$.

find $[\vec{u}.\vec{v}]'$

Sol^n $[u.v]' = \frac{d}{dt}[\vec{u}.\vec{v}] = \vec{u}.\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}.\vec{v}$

$[\sin 2t\,\hat{i} - \cos 2t\,\hat{j} + t\hat{k}].[-2\sin 2t\,\hat{i} - 2\cos 2t\,\hat{j} + 2t\hat{k}] + [2\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j} + \hat{k}].[\cos 2t\,\hat{i} - \sin 2t\,\hat{j} + t^2\hat{k}]$

Now taking dot product.

$= -\frac{1}{2}\sin^2 t + 2\cos^2 2t + 2t^2 + 2\cos^2 2t - 2\sin^2 2t + t^2$. $= 4\cos^2 2t - 4\sin^2 2t + 3t^2 =$

$4[\cos 4t] + 3t^2 = 4\cos 4t + 3t^2$

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Ques: $\vec{u}(t) = 6t^2\hat{i} - t\hat{j} + 3t^2\hat{k}$, $\vec{v}(t) = t\hat{i} + t^2\hat{j} + 2t\hat{k}$.

find $(u \times v)'$

Solⁿ. $(\vec{u} \times \vec{v})' = \frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{dv}{dt} + \frac{d\vec{v}}{dt} \times \vec{v}$

$= (6t^2 - t\hat{j} + 3t^2\hat{k}) \times [\hat{i} + 2t\hat{j} + 2\hat{k}] +$

$[12t\hat{i} - \hat{j} + 6t\hat{k}] \times \{t\hat{i} + t^2\hat{j} + 2t\hat{k}\}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6t^2 & -t & 3t^2 \\ 1 & 2t & 2 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12t & -1 & 6t \\ t & t^2 & 2t \end{vmatrix}$

After Solving

$= (6t^3 - 4t)\hat{i} - \hat{j}(27t^2) + (24t^2 + 2t)\hat{k}$.

or

$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6t^2 & -t & 3t^2 \\ t & t^2 & 2t \end{vmatrix}$

$= \hat{i}[-2t^2 - 3t^4] - \hat{j}\{12t^3 - 3t^3\} +$

$\hat{k}\{6t^4 + t^2\}$

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$$(\vec{u} \times \vec{v}) = \frac{d}{dt}(\vec{u} \times \vec{v}) =$$

$$\hat{i}\left[-6t^2 - 12t^3\right] - \hat{j}\left[27t^2\right] + \hat{k}\cancel{[2t]}$$

$$\hat{k}\left[24t^3 + 2t\right]$$

Ques: Solve $\left[\vec{u}(t) \cdot \left[\vec{u}''(t) \times \vec{u}'(t)\right]\right]'$

$$\left[\vec{u} \cdot (\vec{u}' \times \vec{u}'')\right]' = \frac{d}{dt}\left[\vec{u} \cdot (\vec{u}' \times \vec{u}'')\right]$$

$$= \frac{d}{dt}\left[\vec{u}, \vec{u}', \vec{u}''\right]$$ → scale triple product.

$$= \left[\vec{u}', \vec{u}', \vec{u}''\right] + \left[\vec{u}, \vec{u}'', \vec{u}''\right] + \left[\vec{u}, \vec{u}', \vec{u}'''\right]$$

$$= 0 + 0 + \left[\vec{u}, \vec{u}', \vec{u}'''\right]$$

$$= \left[\vec{u}, \vec{u}', \vec{u}'''\right] \text{ ans}$$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}]$

Scalar triple product vanishes if any two vector are same or proportional

Ques: find parametric eq. of the following curve surfaces.

(i) $x = y,\ y = z$ (ii) $x + y + z = 3,\ y - z = 0$

(iii) $y^2 + z^2 = 9,\ x = 9 - y^2,\ y = 3\sin t$

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Sol^n. (i) $x=y$ $x=y, y=z$

Take $z=t$, $y=z=t$, $x=y=t$.

Parametric eq. is $x=t, y=t, z=t$

(ii) $x+y+z=3$, $y-z=0$

Take $y=t$

$y=z=t$

and $x=3-y-z=3-t-t=3-2t$

$\therefore$ Parametric eq. is $x=3-2t$, $y=t$, $z=t$

(iii) $y^2+z^2=9$, $x=9-y^2$, $y=\sin t$

$y=3\sin t$, $x=9-9\sin^2 t$

$x=9(1-\sin^2 t)$

$x=9\cos^2 t$

Now, $y^2+z^2=9 \Rightarrow z^2=9-y^2$

$=9-9\sin^2 t=9\cos^2 t$

$z=\pm 3\cos t$

Parametric eq is $x=9\cos^2 t$,

$y=3\sin t$

$z=\pm 3\cos t$.

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H.w (1) $y = x^2 + z^2$, $y = 4$, $x = 2\sin t$, Convert into parametric.

(ii) $x - y + z = 5$, $x - 2y + 3z = 3$ (ii).

Ques: find parametric representation of st. line through the point P and has direction of $\vec{b}$.

(1) $P(1,2,3)$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

Sol^n Since, line is parallel to $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$.

D'ration of line $\langle 1:2:2 \rangle$

$\therefore$ eq. of line passing through $P(1,2,3)$ with d'ration $\langle 1:2:2 \rangle$ is

eq. of a str. line in 3-D is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

(x,y,z) is a point on the line & (a,b,c) are dirations of line.

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{2} = t \text{ (say)}$$

$\boxed{x = t+1}$ $\boxed{y = 2t+2}$ $\boxed{z = 2t+3}$ is the parametric eq. of line.

H.w

Ques: find parametric eq. of line pass through $P(1,-1,1)$ & is along the vector $\vec{b} = \hat{i} - \hat{j}$

If $\vec{r} = \vec{r}(t)$ be the eq. of some curve then $\frac{d\vec{r}}{dt}$ represent ~~representation~~ the direction of tangent vector.

Ques: Find parametric eq. of tangent line to the given curve at indicated point

(1) $x = \sin t, \quad y = \cos t, \quad z = t, \quad t = \pi/4$

Sol^n. Since tangent line is touching the curve at $t = \pi/4$, so it is a point on the tangent line.

$t = \pi/4, \quad x = \sin\frac{\pi}{4}, \quad y = \cos\frac{\pi}{2}, \quad z = \frac{\pi}{4}.$

$x = \frac{1}{\sqrt{2}}, \quad y = \frac{1}{\sqrt{2}}, \quad z = \frac{\pi}{4}$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$

$\frac{d\vec{r}}{dt} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

at $\pi/2$

$\left(\frac{dr}{dt}\right)_{t=\pi/4}$ = direction of tangent vector.

$= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$

$\therefore$ D'ratio of tangent line are $\left\langle \frac{1}{\sqrt{2}} : \frac{-1}{\sqrt{2}} : 1 \right\rangle$

$\langle 1 : -1 : -\sqrt{2} \rangle$

$\therefore$ eq. of tangent line to given curve at $t = \frac{\pi}{4}$ {eq. of tangent line to given curve at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ with d'ratios of $\langle 1 : -1 : \sqrt{2} \rangle$} is

$$\frac{x - 1/\sqrt{2}}{1} = \frac{y - 1/\sqrt{2}}{-1} = \frac{z - \pi/4}{\sqrt{2}} = K \text{ (say)}$$

$\therefore$ Parametric eq. of tangent line is

$$x = K + \frac{1}{\sqrt{2}}, \; y = -K + \frac{1}{\sqrt{2}}, \; z = \sqrt{2}K + \frac{\pi}{4}.$$

H.W. Ques: Find parametric eq. of tangent line to the given curve.

(i) $x = t, \; y = 2t^2 \; z = 3t^2, \; t = 2.$

(ii) $x = t^2 - 1, \; y = t + 1, \; z = \frac{t}{t+1} \quad t = 2.$

If $\vec{r} = \vec{r}(t)$ be the eq. of curve, then length of curve from $t = a$ to $t = b$, is

$$L = \int_{t=a}^{b} \left| \frac{d\vec{r}}{dt} \right| dt$$

Ques: Find the length of curve $x = a\cos t$, $y = a\sin t$

$0 \leq t \leq 2\pi$.

Sol: Consider $\vec{r} = x\hat{i} + y\hat{j} = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$\therefore$ length of given curve from $t = 0$ $t = 2\pi$

is $$L = \int_{t=0}^{2\pi} \left| \frac{d\vec{r}}{dt} \right| dt$$

$$= \int_{t=0}^{2\pi} a\,dt = a(t)_{t=0}^{2\pi} = 2\pi a.$$

H.W

Ques: Find the length of curve $\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$, $-2\pi < t < 2\pi$.

Ques: Find the length of curve $\vec{r} = a\cos^3 t\,\hat{i} + a\sin^3 t\,\hat{j}$, $0 < t < \pi/2$.

Ques: Gradient: Let $F(x,y,z)$ be scalar function, then gradient of $F(x,y,z)$ is

$$\text{grad}(F) = \text{grad}(F) = \vec{\nabla} F$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) F$$

$$= \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial F}$$

Gradient is calculated for scalar functions and after gradient of scalar function, we get a vector.

grad F is rate of change of F

Geometrical Interpretation: grad F represent a normal vector to level surface $F(x,y,z) = c$

or

grad F represent a normal vector to level curve, if $(x,y) = c$

Ques: Find gradient of following function at indicated point.

(i) $x^3 - 3x^2y^2 + y^3$, $(1, 2)$

(ii) $\sin(xyz)$ at $(1, -1, \pi)$

(iii) $\log(x^2 + y^2 + z^2)$, $(3, -4, 5)$

(iv) $x^3 + y^3 \sin 4y + z^2$, $(1, \pi/3, 1)$

Sol^n. (i) $f(x,y) = x^3 - 3x^2y^2 + y^3$

$$\text{grad} f = \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\hat{i}\left[3x^2 - 6xy^2\right] + \hat{j}\left[-6x^2y + 3y^2\right] + \hat{k}[0]$$

$$\text{grad} f = \vec{\nabla} f = (3x^2 - 6xy^2)\hat{i} + (-6x^2y + 3y^2)\hat{j}$$

$$\therefore \text{grad} f)_{(1,2)} = (\vec{\nabla} f)_{(1,2)}$$

$$= (3 - 24)\hat{i} + (-12 + 12)\hat{j} = -21\hat{i}$$

(ii) $f(x,y,z) = \sin(xyz)$

$$\text{grad} f = \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\left[\cos(xyz)\,yz \cdot 1\right] + \hat{j}\left[\cos xyz \cdot xz \cdot 1\right]$$

$$\hat{k}\left[\cos(xyz)\,xy \cdot 1\right]$$

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$\vec{\nabla} F = yz \cos(xyz)\hat{i} + xz\cos xyz\hat{j} + xy\cos xyz\hat{k}$

$\vec{\nabla} F_{(1,-1,\pi)} = -\pi\cos(-\pi)\hat{i} + \pi\cos(-\pi)\hat{j} - \cos(-\pi)\hat{k}$

$\vec{\nabla} F(1,-1,\pi) = \pi\hat{i} - \pi\hat{j} + \hat{k}$

Ques: find normal vector and unit normal vector to the given curl surface at the indicated point.

(i) $y^2 = 16x$ $(4,8)$ (ii) $x^2+2y^2+z^2=4$ $(1,1,1)$

(iii) $z = xy$, $(-1,-2,2)$ (iv) $z^2 = x^2 - y^2$ $(2,1,\sqrt{3})$

Solⁿ (i) Eq. of curve $y^2 = 16x$

$y^2 - 16x = 0$

$\vec{\nabla} F = \text{normal}$

$F(x,y) = C$

Here, $F(x,y) = y^2 - 16x$

$\therefore$ Normal to the given curve (i) is $\vec{\nabla} F$

$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}(-16) + \hat{j}(2y) + \hat{k}(0)$

$= -16\hat{i} + 2y\hat{j}$

$\therefore$ Normal to the curve (i) at $(4,8) = (\vec{\nabla} F)_{(4,8)}$

$= -16\hat{i} + 16\hat{j} = \vec{N}$

∴ Unit normal to the curve (1) at (4,8)

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{-16\hat{i} + 16\hat{j}}{\sqrt{(-16)^2 + (16)^2}}$$

$$= \frac{-16(\hat{i} - \hat{j})}{16\sqrt{2}}$$

$$= \frac{-\hat{i} + \hat{j}}{\sqrt{2}}$$ → unit normal vector

or

$$\frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

both are corect. → {aware in mcq}

(ii) Eq. of surface is $x^2 + 2y^2 + z^2 = 4$

$\Rightarrow x^2 + 2y^2 + z^2 - 4 = 0$ — (1)

Hence $(f(x,y,z)) = x^2 + 2y^2 + z^2 - 4$.

∴ Normal to the surface (1) is $\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$

$= \hat{i}(2x) + \hat{j}(4y) + \hat{k}(2z)$

∴ Normal to surface (1) at (1,1,1) =

$$(\vec{\nabla} f)_{(1,1,1)}$$

$= 2\hat{i} + 4\hat{j} + 2\hat{k} = \vec{N}$.

∴ unit normal to surface (I) at (1,1,1)

$$= \frac{\vec{N}}{|\vec{N}|} = \frac{2\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{4+16+4}}$$

$$= \frac{2[\hat{i} + 2\hat{j} + \hat{k}]}{2\sqrt{6}} = \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

Ques: Find the eq. of tangent plane to the graph of the equation.

(I) $x^2 - 3y^2 - z^2 = 2$, $(3,1,2)$

(II) $z = 16 - x^2 - y^2$ $(1,3,6)$

Sol.

Since, we need to find eq. of tangent plane to surface $x^3 - 3y^2 - z^2 = 2$ at $(3,1,2)$ so we know one point on the tangent plane is $(3,1,2)$

Eq. of plane in 3-D is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ where (x_1,y_1,z_1) is point on plane & $\langle a,b,c \rangle$ are direction of normal to plane.

Eq. of surface $x^3 - 3y^2 - z^2 = 2$

$$\Rightarrow x^3 - 3y^2 - z^2 = 0$$

$$F(x,y,z) = x^2 - 3y^2 - z^2 - 2.$$

Normal to the given surface is =

$$\vec{\nabla} F = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$= \hat{i}[2x] + \hat{j}[-6y] + \hat{k}[-2z]$

$\therefore$ Normal to the surface at $(3,1,2)$

$= (\vec{\nabla} F)_{(3,1,2)}$

$= 6\hat{i} - 6\hat{j} - 4\hat{k}$

D' ratio of normal to surface at $(3,1,2)$ are

$\langle 6, -6, -4 \rangle$, $\langle 3, -3, -2 \rangle$

Eq. of tangent plane to the given surface, which passes through $(3,1,2)$ with direction of normal to plane are $\langle 3, -3, -2 \rangle$

$3(x-3) - 3(y-1) - 2(z-2) = 0$

$\Rightarrow 3x - 9 - 3y + 3 - 2z + 4 = 0$

$\Rightarrow$ ~~$3x - 3y - 2z + 4 = 0 \Rightarrow 3x - 3y - 2z - 2 = 0$~~

$\Rightarrow 3x - 3y - 2z - 2 = 0$

Ques: Find the angle between the surface at the indicated point

(i) $z = x^2 + y^2$, $z = 2x^2 - 3y^2$ $(2,1,5)$

(ii) $x^2 + y^2 = 4$, $x^2 + y^2 + z^2 = 12$, $(2,2,-2)$

sol. (1) Eq. of first surface $z = x^2+y^2$ =>

$$x^2+y^2-z=0$$

let $f_1(x,y,z) = x^2+y^2-z$

Normal to first surface: $\nabla f_1 = \hat{i}\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} + \hat{k}\frac{\partial f_1}{\partial f}$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-1)$$

∴ Normal to the first surface at $(2,1,5)$

$$= \vec{N_1} = (\vec{\nabla} f_1)_{(2,1,5)}$$

$$\vec{N_1} = 4\hat{i} + 2\hat{j} - \hat{k}$$

Again, eq. of 2nd surface is $z = 2x^2 - 3y^2$

$$\Rightarrow 2x^2 - 3y^2 - z = 0$$

$$f_2(x,y,z) = 2x^2 - 3y^2 - z$$

Normal to 2nd surface is $= \vec{\nabla} f = \hat{i}\frac{\partial f_2}{\partial x} + \hat{j}\frac{\partial f_2}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$= \hat{i}[4x] + \hat{j}[-6y] + \hat{k}(-1)$$

∴ Normal to 2nd surface at $(2,1,5)$

$$= (\vec{\nabla} f_2)_{(2,1,5)} = \vec{N_2}$$

$$\vec{N_2} = 8\hat{i} - 6\hat{j} - \hat{k}$$

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Angle b/w two surface at (2,1,5) is same as the angle b/w their normals at (2,1,5)

$\therefore \vec{N_1}\cdot\vec{N_2} = |\vec{N_1}||\vec{N_2}| \cos\theta$ where θ is the angle b/w $\vec{N_1}, \vec{N_2}$

$$\Rightarrow \cos\theta = \frac{\vec{N_1}\cdot\vec{N_2}}{|\vec{N_1}||\vec{N_2}|} = \frac{32+(-12)+(1)}{\sqrt{4^2+2^2+(-1)^2}\sqrt{8^2+(6)^2+(-1)^2}}$$

$$= \frac{21}{\sqrt{21}\sqrt{101}} = \frac{\sqrt{21}}{\sqrt{101}} = \sqrt{\frac{21}{101}}$$

$$\theta = \cos^{-1}\left[\frac{\sqrt{21}}{101}\right]$$

Ques: Find the parametric eq. of the normal line at the given point.

(i) $z = 3x^2 - 2y^2$, at $(2,1,10)$

(ii) $x^2 + 2y^2 + 4z^2 = 10$, $(2,1,-1)$

Soln (i) Eq. of surface

$z = 3x^2 - 2y^2$

$\Rightarrow 3x^2 - 2y^2 - z = 0$

Eq. of line (1) is is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

(x_1, y_1, z_1) is a point on line. $\langle a, b, c\rangle$ are d-ratio of the line.

Here $f(x,y,z) = 3x^2 - 2y^2 - z$

Normal to the given surface is $\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}(6x) + \hat{j}(-4y) + \hat{k}(-1)$

$\therefore$ Normal to the given surface at $(2,1,10)$ is

$$(\vec{\nabla} f)_{(2,1,10)} = 12\hat{i} - 4\hat{j} - \hat{k}$$

$\therefore$ D'ratio of normal are $\langle 12 : -4 : -1 \rangle$

$\therefore$ Eq. of normal line to the given surface at $(2,1,10)$ with d-ratio of normal are $\langle 12 : -4 : -1 \rangle$ is
$\langle a, b, c \rangle$

$$\frac{x-2}{12} = \frac{y-1}{-4} = \frac{z-10}{-1} = t \text{ (say)}$$

$x = 12t + 2,\ y = -4t + 1,\ z = t + 10$ is the parametric eq. of normal line.

$\Rightarrow$ Directional Derivative:

Let $f(x,y,z)$ be a scalar function, then directional derivative of $f(x,y,z)$ in the direction of $\vec{b}$ is given as $(\vec{\nabla} f) \cdot \hat{b}$ where $\hat{b}$ is unit vector along $\vec{b}$.

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Directional derivative give us rate of change of f in the direction of b.

Maximum rate of change of any scalar valued function $f(x,y,z)$ is $|\vec{\nabla} f|$ and it occurs in the direction of $\vec{\nabla} f$.

Minimum rate of change of any scalar function $f(x,y,z)$ is $-|\vec{\nabla} f|$ and it occurs in the direction of $-(\nabla f)$ or opposite to $(\vec{\nabla} f)$.

Ques: find directional derivative of given scalar function at given point in the indicated direction.

(i) xyz, $(1,4,3)$ in the direction of line from $(1,2,3)$ to $(4,-1,-3)$.

(ii) $\sqrt{xy^2 + 2x^2z}$, $(2,-2,+1)$ in the direction of $-ve$ z-axis.

(iii) $x^2y - y^2z - xyz$, $(1,-1,0)$ in the direction of $\hat{i} - \hat{j} + 2\hat{k}$

sol. (i) Let $f(x,y,z) = xyz$

$$\vec{\nabla} f = \text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$\hat{i}(yz \cdot 1) + \hat{j}[xz \cdot 1] + \hat{k}[xy \cdot 1]$

$(\vec{\nabla} f)_{(1,4,3)} = 12\hat{i} + 3\hat{j} + 4\hat{k}$

Vector point along a line starting with $(1, 2, 3)$ upto $\{1, -1, -3\}$

$= \vec{b}$ = position vector of Q − P.V of P

$= (\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= -3\hat{j} - 6\hat{k}$

Now, $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{-3\hat{j} - 6\hat{k}}{\sqrt{9+36}} = \dfrac{-3[\hat{j} + 2\hat{k}]}{3\sqrt{5}}$

$= \dfrac{-[\hat{j} + 2\hat{k}]}{\sqrt{5}}$

Directional Derivative of (x, y, z) at $(1, 4, 3)$ in the direction of $\vec{b}$ is $(\vec{\nabla} f) \cdot \vec{b}_{(1,4,3)}$

$= -(12\hat{i} + 3\hat{j} + 4\hat{k}) \cdot \left[\dfrac{\hat{j} + 2\hat{k}}{\sqrt{5}}\right]$

$= \dfrac{-3-8}{\sqrt{5}} = \dfrac{-11}{\sqrt{5}}$

(11) $F(x,y,z) = \sqrt{xy^2+2x^2z}$

Consider $\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$

$$= \hat{i}\,\frac{1}{2}\left[xy^2+2x^2z\right]^{-1/2}\left[y^2+4xz\right] +$$

$$\hat{j}\left[\frac{1}{2}\left(xy^2+2x^2z\right)^{1/2}\left[2xy\right]\right] +$$

$$\hat{k}\cdot\frac{1}{2}\left[xy^2+2x^2z\right]^{1/2}\left[2x^2\right]$$

$$= \frac{(y^2+4xz)\hat{i} + 2xy\,\hat{j} + 2x^2\,\hat{k}}{2(xy^2+2x^2z)^{1/2}}$$

$$\therefore (\vec{\nabla}F)_{(2,-2,1)} = \frac{12\hat{i} - 8\hat{j} + 8\hat{k}}{2\sqrt{8+8}}$$

$$= \frac{4\left[3\hat{i} - 2\hat{j} + 2\hat{k}\right]}{2\times 4}$$

$$= \frac{3\hat{i} - 2\hat{j} + 2\hat{k}}{2}$$

Now, unit vector going in the direction of negative z-axis. $-\hat{k}$

$\therefore$ D.D of $f(x,y,z)$ at $(2,-2,1)$ along

~~neg~~

negative z-axis is $(\vec{\nabla}f)_{(2,-2,1)} \cdot (-\hat{k})$

$$= \frac{-2}{2} = -1$$

<u>Ques</u>: find a vector that gives maximum rate of increase. find the maximum rate.

(i) $e^{2y} \cos x, \left(\frac{\pi}{4}, 0\right)$

(ii) $3x^2 + y^2 + 2z^2$ at $(0,1,2)$

(iii) $6xyz$ at $(-1,2,1)$

(iv) $x^2y^2z^2 + xz^2 + x^2y$, $(1,2,-1)$

<u>sol.</u> $f(x,y,z) = x^2y^2z^2 + xz^2 + x^2y$

$$\text{Grad} f = \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\{2xy^2z^2 + z^2 + 2xy\} + \hat{j}\{2x^2yz^2 + x^2\}$$

$$+ \hat{k}\{2x^2y^2z + 2xz\}$$

$$\therefore (\vec{\nabla} f)_{(1,2,-1)} = \hat{i}\{8 + 1 + 4\} + \hat{j}\{4 + 1\} +$$

$$\hat{k}\{-8 - 2\} = 13\hat{i} + 5\hat{j} - 10\hat{k}$$

Now, maximum of rate of increase/change of $f(x,y,z)$ at $(1,2,-1)$

$$= \left| (\vec{\nabla} F)_{(1,2,-1)} \right|$$

$$= \sqrt{169 + 25 + 100}$$

$\therefore$ The vector along the maximum rate of increase / change of $F(x,y,z)$ at $(1,2,-1) = (\vec{\nabla} F)(1,2,-1) = 13\hat{i} + 5\hat{j} - 10\hat{k}$

Ques: Find the vector that gives the direction of minimum rate of increase. Find the minimum rate.

(i) $x^3 - xy^2 + y^2$ $(-2,-1)$

(ii) $x^2 - y^2 + z^2$ $(1,2,1)$

Soln: let $F(x,y,z) = x^2 - y^2 + z^2$

$$\vec{\nabla} F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(-2y) + \hat{k}(2z)$$

$$\therefore (\vec{\nabla} F)_{(1,2,1)} = 2\hat{i} - 4\hat{j} + 2\hat{k}$$

$\therefore$ Minimum rate of increase / change of $F(x,y,z)$ at $(1,2,1) = -|\vec{\nabla} F|$

$$= -\sqrt{4+16+4} = -2\sqrt{6}$$

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$\therefore$ Vector in direction of minimum rate of increase/change of $f(x,y,z)$ at $(1,2,1)$ =

$$-\vec{\nabla}(1,2,1) = -2\hat{i}+4\hat{j}-2\hat{k}$$

Ques: Find a scalar function f such that $\vec{v} = \vec{\nabla}F$ where $\vec{v} = xy\{2yz\hat{i} + 2xz\hat{j} + xy\hat{k}]$

Soln: Since $\vec{v} = \vec{\nabla}F$ | scalar function gradient ↓ vector function.

$$\Rightarrow 2xy^2z\hat{i} + 2x^2yz\hat{j} + x^2y^2\hat{k}$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial x} = 2xy^2z}_{\text{integrate with w.r.t } x} \quad (i) \qquad \frac{\partial f}{\partial y} = 2x^2yz \quad (ii)$$

$$\frac{\partial f}{\partial z} = x^2y^2 \quad (iii)$$

$$f(x,y,z) = \frac{2x^2y^2z}{2} + \phi(y,z) \quad (A) \leftarrow \text{put in (ii)}.$$

(ii) $\frac{\partial}{\partial y}[x^2y^2z + \phi(y,z)] = 2x^2yz$

$$\Rightarrow 2x^2yz + \frac{\partial \phi}{\partial y} = 2x^2yz$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = 0$$

Integrate w.r.t y.

$\phi(y,z) = K(z)$ ← put in ☆

$\therefore F(x,y,z) = x^2y^2z + K(z)$ — ☆☆ put in (iii)

$$\frac{\partial}{\partial z}\left[x^2y^2z + K(z)\right] = x^2y^2$$

$$x^2y^2 + \frac{dK}{dz} = x^2y^2$$

$\frac{dK}{dz} = 0$ → Integrate w.r.t z

$$K(z) = C$$

$$\boxed{F(x,y,z) = \cancel{x^4+y^3}\ x^2y^2z + C}$$

verification : verify above question at your own

Ques: find a scalar valued function F such that $\vec{v} = \vec{\nabla} F$.

where $\vec{v} = 12x\hat{i} - 15y^2\hat{j} + \hat{k}$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 12x \text{ — (i)} \quad \frac{\partial f}{\partial y} = -15y^2 \text{ — (ii)} \quad \frac{\partial f}{\partial z} = 1 \text{ — (iii)}$$

Integration w.r.t x.

Integrate w.r.t x.

$$F(x,y,z) = 6x^2 + \phi(y,z) \text{ — } \star$$

$$\text{(ii)} \Rightarrow \frac{\partial}{\partial y}[6x^2 + \phi(y,z)] = -15y^2$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = -15y^2$$

Integrating w.r.t y.

$$\phi(y,z) = -5y^3 + K(z) \text{ Put in } \star$$

$$\star \Rightarrow f(x,y,z) = 6x^2 - 5y^3 + K(z) \text{ — } \star\star$$

Put in (iii)

$$\text{(iii)} \quad \frac{\partial}{\partial z}[6x^2 - 5y^3 + K(z)] = 1$$

$$\Rightarrow \frac{\partial K}{\partial z} = 1$$

Integrate w.r.t. z

$K(z) = z + C$ put in ★★

$F(x,y,z) = 6x^2 - 5y^3 + z + C$

Divergence : Let $\vec{v}(x,y,z)$ be a vector function, then divergence of $\vec{v}(x,y,z) =$

$$\text{div}(\vec{v}) = \text{Div}(\vec{v}) = \vec{\nabla}\cdot\vec{v} = \hat{i}\cdot\frac{\partial\vec{v}}{\partial x} + \hat{j}\frac{\partial\vec{v}}{\partial y} + \hat{k}\frac{\partial\vec{v}}{\partial z}$$

$$= \underbrace{\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}_{}$$ where $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

↳ divergence of $\vec{v}$ as scalar function.

$\vec{\nabla}\cdot\vec{v} \neq \vec{v}\cdot\vec{\nabla}$

Divergence ↑ ↑ Operator

Solenoidal vector : If $\text{div}\,\vec{v} = 0$, then $\vec{v}$ is solenoidal vector.

Curl : Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be a vector function then.

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$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \hat{i} \times \frac{\partial v}{\partial x} + \hat{j} \times \frac{\partial \vec{v}}{\partial y} + \hat{k} \times \frac{\partial v}{\partial z}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\hat{i}\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] - \hat{j}\left[\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right] + \hat{k}\left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

curl $\vec{v}$ is a vector function.

Irrotational vector: If curl $\vec{v} = 0$

$\Rightarrow \vec{v}$ is irrotational vector.

Formula: (i) curl(grad ϕ) = 0, where ϕ is a sclar function.

(ii) div [curl $\vec{v}$] = 0, where $\vec{v}$ is a vector function.

Ques: find div $\vec{v}$, curl $\vec{v}$, curl $\vec{v}$ & show that div[curl $\vec{v}$] = 0.

(i) $(x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$

(ii) $xe^{-y}\hat{i} + 2ze^{-y}\hat{j} + xy^2\hat{k}$.

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(iii) $xyz\,\hat{i} + 2x^2y\,\hat{j} + (xz^2 - y^2z)\,\hat{k}$

Sol. (ii) Let $\vec{v} = xe^{-y}\hat{i} + 2ze^{-y}\hat{j} + xy^2\hat{k}$

$$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(xe^{-y}) + \frac{\partial}{\partial y}(2ze^{-y}) + \frac{\partial}{\partial z}(xy^2)$$

$$= e^{-y} + 2ze^{-y}(-1) + 0$$

$$\text{div}\,\vec{v} = e^{-y}[1-2z]$$

Now, $\text{Curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & 2ze^{-y} & xy^2 \end{vmatrix}$

$$= \hat{i}[2xy - 2e^{-y}] - \hat{j}[y^2 - 0] + \hat{k}[0 - xe^{-y}(-1)]$$

$$\vec{\nabla}\times\vec{v} = (2xy - 2e^{-y})\hat{i} - y^2\hat{j} + xe^{-y}\hat{k}.$$

Now, $\text{div}[\text{curl}\,\vec{v}] = \vec{\nabla}\cdot[\vec{\nabla}\times\vec{v}] =$

$$\frac{\partial}{\partial x}[2xy - 2e^{-y}] + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(xe^{-y})$$

$$= 2y - 0 - 2y + 0 = 0$$

Ques: find grad f, and hence verify that

Curl (grad f) = 0.

(i) $f = x + y - 2z^2$ (ii) e^{x+y+z}

(iii) $16xy^3z^2$ (iv) $x\sin(x+y+z)$

sol^n (iii). $\vec{\nabla} f = \text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial z} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}[16y^3z^2] + \hat{j}[48xy^2z^2] + \hat{k}[32xy^3z]$

Now, Curl (grad f) $= \vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 16y^3z^2 & 48xy^2z^2 & 32xy^3z \end{vmatrix}$

$= \hat{i}[96xy^2z - 96xy^2z] - \hat{j}[32y^3z - 32y^3z]$

$+ \hat{k}[48y^2z^2 - 48y^2z^2]$

Curl (grad f) = 0

Ques: show that following vectors are solenoidal

(i) $(2x+3y)\hat{i} + (x-y)\hat{j} - (x+y+z)\hat{k}$

(ii) $e^{x+y-2z}(\hat{i}+\hat{j}+\hat{k})$

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sol- (ii) let $\vec{v} = e^{x+y-2z}\hat{i} + e^{x+y-2z}\hat{j} + e^{x+y-2z}\hat{k}$.

Now, $\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\left[e^{x+y-2z}\right] +$

$$\frac{\partial}{\partial y}\left[e^{x+y-2z}\right] + \frac{\partial}{\partial z}\left[e^{x+y-2z}\right]$$

$$= e^{x+y-2z}\cdot 1 + e^{x+y-2z} + e^{x+y-2z}(-2)$$

$$\text{div}\,\vec{v} = 0$$

$\Rightarrow \vec{v}$ is solenoidal.

Ques: Find curl $\vec{r}$ and div $\vec{r}$ where

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

sol: Curl $\vec{r} = \vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0]$$

$$= \vec{0}$$

$\vec{r}$ is irrotational.

Now, $\text{div}\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}z$

$$= 1 + 1 + 1 = 3$$

Ques: find the value of a, b, c such that

$$\vec{v} = (3x + ay + z)\hat{i} + (2x - y + bz)\hat{j} + (x - cy + 7z)\hat{k}.$$

Sol. Since $\vec{v}$ is irrotional $\Rightarrow \vec{\nabla}\times\vec{v} = \text{Curl}\,\vec{v} = 0$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x+ay+z & 2x-y-bz & x+cy+7z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \hat{i}[c-b] - \hat{j}[1-1] + \hat{k}[2-a] = \vec{0}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

$\boxed{c-b=0}$ $\boxed{2-a=0}$ $\boxed{a=2}$

$\boxed{b=c}$

c can take any value

Ques: If $f = x^2+y^2+z^2$ & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

show that $\text{div}[f\vec{r}] = 5f$

Sol. $f\vec{r} = (x^2+y^2+z^2)[x\hat{i} + y\hat{j} + z\hat{k}]$

$$= (x^3 + xy^2 + xz^2)\hat{i} + (x^2y + y^3 + yz^2)\hat{j} + (x^2z + y^2z + z^3)\hat{k}$$

$$\text{div}\,[f\vec{r}] = \vec{\nabla}\cdot[f\vec{r}]$$

$$= \frac{\partial}{\partial x}[f\vec{r}] = \frac{\partial}{\partial x}[x^3 + xy^2 + xz^2] + \frac{\partial}{\partial y}[x^2y + y^3 + yz^2] + \frac{\partial}{\partial z}(x^2z + y^2z + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2$$

$$= 5x^2 + 5y^2 + 5z^2 = 5[x^2 + y^2 + z^2] = 5f$$

* formula :

(iii) $\text{div}[\phi\vec{u}] = \text{grad}\,\phi\cdot\vec{u} + \phi\,\text{div}\,\vec{u}$, where ϕ is sclar function & $\vec{u}$ is vector function.

$$\Rightarrow \vec{\nabla}\cdot[\phi\vec{u}] = \nabla\phi\cdot\vec{u} + \phi(\vec{\nabla}\cdot\vec{u})$$

(iv) $\text{curl}\,[\phi\vec{u}] = \text{grad}\,\phi\times\vec{u} + \phi\,\text{curl}\,\vec{u}$

$$\Rightarrow \vec{\nabla}\times[\phi\vec{u}] = \vec{\nabla}\phi\times\vec{u} + \phi(\vec{\nabla}\times\vec{u})$$

(v) $\text{div}\,[\vec{u}\times\vec{v}] = \vec{v}\cdot\text{curl}\,\vec{u} - \vec{u}\cdot\text{curl}\,\vec{v}$

$$= \nabla\cdot[\vec{u}\times\vec{v}] = \vec{v}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot[\vec{\nabla}\times\vec{v}]$$

(vi) $\text{curl}\,(\vec{u}\times\vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{v} + \vec{u}(\vec{\nabla}\cdot\vec{v}) - \vec{v}(\vec{\nabla}\cdot\vec{u})$

(vii) $\text{grad}(\vec{u}\cdot\vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{v}$
$+ \vec{v}\times \text{curl}\,\vec{u} + \vec{u}\times \text{curl}\,\vec{v}$.

(viii) $\text{Curl}(\text{curl}\,\vec{v}) = \text{grad}(\text{div}\,\vec{v}) - \nabla^2\vec{v}$

$\vec{\nabla}[\vec{\nabla}\times\vec{v}] = \vec{\nabla}[\vec{\nabla}\cdot\vec{v}] - \nabla v^2,$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, Lapacian operator.

Ques: If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then prove that

(i) $\text{grad}(\vec{a}\cdot\vec{r}) = \vec{a}$ (ii) $\text{div}[\vec{a}\times\vec{r}] = 0$

(iii) $\text{curl}(\vec{a}\times\vec{r}) = 2\vec{a}$ (iv) $\text{div}[(\vec{a}\cdot\vec{r})\vec{r}] = 4(\vec{a}\cdot\vec{r})$

Sol. (i)

(i) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a constant vector.

Now, $\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

$\text{grad}(\vec{a}\cdot\vec{r}) = \vec{\nabla}(\vec{a}\cdot\vec{r}) = \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$

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$= \underbrace{\hat{i}(a_1) + \hat{j}(a_2) + (a_3)\hat{z}}_{\vec{a}}$

$\text{grad}(\vec{a}\cdot\vec{r}) = \vec{a}$

(ii) $\text{div}[\vec{a}\times\vec{r}] =$ | $\text{div}[\vec{u}\times\vec{v}] =$

$\vec{r}\cdot \text{Curl}\,\vec{a} - \vec{a}\cdot \text{Curl}\,\vec{r}$ | $\vec{v}\cdot \text{Curl}\,\vec{u} - \vec{u}\cdot \text{Curl}\,\vec{v}$

— (1)

Now, $\text{Curl}(\vec{a}) = \vec{\nabla}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0] = \vec{0}$.

$\text{Curl}\,\vec{r} = \vec{0}$.

$\text{div}[\vec{a}\times\vec{r}] = \vec{r}\cdot\vec{0} - \vec{a}\cdot\vec{0} = 0$.

(iii) $\text{Curl}(\vec{a}\times\vec{r}) = 2\vec{a}$

Here, $\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$= \hat{i}[a_2 z - a_3 y] - \hat{j}[a_1 z - a_3 x] + \hat{k}[a_1 y - a_2 x]$

Now, $\text{Curl}(\vec{a}\times\vec{r}) = \vec{\nabla}\times(\vec{a}\times\vec{r})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \hat{i}\{a_1 - (-a_1)\} - \hat{j}\{-a_2 - a_2\} + \hat{k}\{a_3 - (-a_3)\}$$

$$2a\hat{i} + 2a_2\hat{j} + 2a_3\hat{k} = 2\{a\hat{i} + a_2\hat{j} + a_3\hat{k}\}$$

$$= 2\vec{a}.$$

(IV) divergence ~~of~~

$$\operatorname{div}\{(\vec{a}\cdot\vec{r})\vec{r}\} = 4\{\vec{a}\cdot\vec{r}\}$$

Consider $\operatorname{div}\{\underbrace{\vec{a}\cdot\vec{r}}_{\phi})\underbrace{\vec{r}}_{\vec{u}}\} = \nabla\cdot\{(\vec{a}\cdot\vec{r})\vec{r}\}$

$\operatorname{div}\{\phi\vec{u}\} =$

$\operatorname{grad}\phi\cdot\vec{u} + \phi\operatorname{div}\vec{u}$

$$\operatorname{grad}(\vec{a}\cdot\vec{r})\cdot\vec{r} + (\vec{a}\cdot\vec{r})\operatorname{div}(\vec{r})$$

$$= \vec{a}\cdot\vec{r} + 3(\vec{a}\cdot\vec{r})$$ | By using previous question.

$$4(\vec{a}\cdot\vec{r}).$$

Ques: If $\vec{E}$ & $\vec{H}$ are irrotational vector, then show that $\vec{E} \times \vec{H}$ is solenoidal.

Sol. Since $\vec{E}, \vec{H}$ are irrotational $\Rightarrow$ curl $\vec{E} = \vec{\nabla} \times \vec{E}$

$$\text{curl } \vec{E} = \vec{\nabla} \times \vec{E} = 0$$

$$\& \text{ curl } \vec{H} = \vec{\nabla} \times \vec{H} = 0$$

Now, $\text{div}[\vec{E} \times \vec{H}] = \vec{\nabla}\{\vec{E} \times \vec{H}\}$

$\{ \text{div}[\vec{u} \times \vec{v}] = \vec{v} \cdot \text{curl } \vec{u} - \vec{u} \cdot \text{curl } \vec{v}$.

$H \cdot \text{curl } \vec{E} - \vec{E} \cdot \text{curl } \vec{H} = \vec{H} \cdot \vec{0} - \vec{E} - \vec{0}] = 0$

$= \vec{E} \times \vec{H}$ is solenoidal.

Ques: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that $(\vec{u} \cdot \vec{\nabla})\vec{r} = \vec{u}$.

Sol^n. let $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ &

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

$$(\vec{u} \cdot \vec{\nabla}) = \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}\right)$$

Now, $(\vec{u} \cdot \vec{\nabla})\vec{r} = u_1\frac{\partial}{\partial x}\vec{r} + u_2\frac{\partial}{\partial y}\vec{r} + u_3\frac{\partial}{\partial z}\vec{r}$

$$= u_1[\hat{i} + 0 + 0] + u_2(\hat{j}) + u_3(\hat{k})$$

$$(\vec{u} \cdot \vec{\nabla})\vec{r} = \vec{u}.$$